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Noncommutative Scalar Solitons: Existence and Nonexistence

Bergfinnur Durhuus[a1], Thordur Jonsson[b23] and Ryszard Nest[a4]

[a]Matematisk Institut, Universitetsparken 5
2100 Copenhagen Ø, Denmark

[b]The Niels Bohr Institute and NORDITA, Blegdamsvej 17
2100 Copenhagen Ø, Denmark

Abstract. We study the variational equations for solitons in noncommutative scalar field theories in an even number of spatial dimensions. We prove the existence of spherically symmetric solutions for a sufficiently large noncommutativity parameter θ and we prove the absence of spherically symmetric solutions for small θ.

[1]email: durhuus@math.ku.dk
[2]e-mail: thjons@raunvis.hi.is
[3] Permanent address: Science Institute, University of Iceland, Dunhaga 3, 107 Reykjavik, Iceland
[4]email: rnest@math.ku.dk

1 Introduction

Recently there has been considerable interest in solitons in noncommutative field theories, the main motivation coming from string theory [1, 2]. Several authors have found explicit solitons in gauge theories with and without matter fields [3, 4, 5, 6, 7].

In [8] solitons in scalar field theories were studied and it was shown that there is a host of solitons in the case of an infinite noncommutativity parameter θ where the kinetic term in the action can be neglected. This is in a stark contrast to the commutative case where no solitons exist [9]. The authors of [8] conjectured the existence of solitonic solutions at large θ and indicated how to calculate them perturbatively in θ^{-1}. In [10] the existence problem for solitons at finte θ was discussed and it was argued that solitons would not exist for small θ. Various aspects of solitons in noncommutative scalar field theories are discussed in [11, 12, 13, 14, 15].

In this paper we establish the existence of spherically symmetric solitons in even dimensional scalar field theories under fairly general conditions on the potential, provided θ is sufficiently large. We show that no spherically symmetric solutions can exist for small θ. We briefly comment on the problem of generalizing these results to the nonrotationally invariant case. In the bulk of the paper we deal, for simplicity, with the two-dimensional case and describe in the last section the changes needed to treat the general even dimensional case.

2 The problem

Solitons in a noncommutative two-dimensional scalar field theory with potential V are defined as finite energy solutions to the variational equations of the action functional

$$S(\varphi) = \int \left(\frac{1}{2}(\nabla\varphi)^2 + V_\theta(\varphi)\right) d^2x. \tag{1}$$

Here the potential V applied to the function φ should be calculated using the Moyal product $\star$ on the space of functions on $\mathbf{R}^2$. If V is a polynomial we can express $V_\theta(\varphi)$ as

$$V_\theta(\varphi) = \sum_{j=1}^{n} c_j \varphi^{\star j} \tag{2}$$

where the star powers are given by the star product

$$(\varphi \star \psi)(x) = \exp\left(i\frac{\theta}{2}\varepsilon_{jk}\frac{\partial}{\partial y_j}\frac{\partial}{\partial z_k}\right)\varphi(y)\psi(z)\Big|_{z=y=x} \tag{3}$$

and we sum over repeated indices. Here ε_{jk} is the antisymmetric symbol, $\varepsilon_{12} = -\varepsilon_{21} = 1$, and $\theta > 0$ is the noncommutativity parameter. We assume that $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$.

The algebra of smooth rapidly decaying functions on $\mathbf{R}^2$ with the Moyal product defined above is well known to be isomorphic to an algebra of operators on a one particle Hilbert space. We denote this isomorphism by $\varphi \mapsto \hat{\varphi}$. Under the isomorphism the $\star$-product becomes the usual operator product and

$$\frac{1}{2\pi}\int \varphi(x)\, d^2x = \theta \mathrm{Tr}\hat{\varphi}. \tag{4}$$

For a discussion of this correspodence, see, e.g., [8, 16]. In the operator formalism the action functional becomes

$$S(\hat{\varphi}) = \mathrm{Tr}\left([a,\hat{\varphi}][\hat{\varphi},a^*] + \theta V(\hat{\varphi})\right), \tag{5}$$

where a^* and a are the usual raising and lowering operators of the simple harmonic oscillator and the operator $\hat{\varphi}$ has been assumed to be self-adjoint which corresponds to a real valued function φ. The variational equation of (5) is

$$2[a,[a^*,\hat{\varphi}]] + \theta V'(\hat{\varphi}) = 0. \tag{6}$$

If we choose the harmonic oscillator eigenstates as the basis for our Hilbert space then it is easily checked that radially symmetric functions φ correspond to diagonal operators $\hat{\varphi}$. If we consider a diagonal operator with eigenvalues λ_n, $n = 0, 1, 2, \ldots$, Eq. (6) reduces to [8, 10]

$$\begin{aligned} (n+1)\lambda_{n+1} - (2n+1)\lambda_n + n\lambda_{n-1} &= \frac{\theta}{2}V'(\lambda_n), \quad n \geq 1 && (7)\\ \lambda_1 - \lambda_0 &= \frac{\theta}{2}V'(\lambda_0). && (8) \end{aligned}$$

We can sum the second order finite difference equation for λ_n from $n = 0$ to $n = N$ and obtain the first order equation

$$\lambda_{N+1} - \lambda_N = \frac{\theta}{2(N+1)}\sum_{n=0}^{N} V'(\lambda_n), \quad N \geq 0. \tag{9}$$

A necessary condition for the energy to be finite is clearly that

$$\lambda_N \to 0 \quad \text{as} \quad N \to \infty. \tag{10}$$

The first problem we wish to address is when there do exist solutions to Eq. (9) satisfying the boundary condition (10).

3 Properties of solutions

In this section we derive some properties that any solution to Eqs. (9) and (10) must satsfy under some simple conditions on the potential V. We assume that V is twice continuously differentiable and has only one local minimum in addition to 0. Let the other local minum be at $s > 0$. Let $r \in (0, s)$ be a point where V has a local maximum and for technical convenience assume that V' does not vanish except at $0, r$ and s. Then $V'(x) < 0$ for $x < 0$ or $x \in (r, s)$ and $V'(x) > 0$ for $x > s$ or $x \in (0, r)$.

Let us now assume that λ_n is a sequence of real numbers which solves Eq. (9) with the boundary condition (10). We assume that the λ_n's are not all zero. In this case we have the following:

(a) $0 < \lambda_n < s$, for all n.

(b) λ_n tends monotonically to 0 for n large enough

(c) $\sum_n V'(\lambda_n) = 0$ and $\sum_n \lambda_n < \infty$.

We now establish these three properties in turn.

Suppose $\lambda_0 \geq s$. Then $V'(\lambda_0) \geq 0$ so by Eq. (9) $\lambda_1 \geq \lambda_0$ and λ_n is a nondecreasing sequence by induction. Similarly, if $\lambda_0 < 0$ then λ_n is a decreasing sequence. Hence, $0 \leq \lambda_0 < s$. If $\lambda_n \notin (0, s)$ for some values of n, let $p > 0$ be the smallest such n. If $\lambda_p \geq s$ then $\lambda_p - \lambda_{p-1} > 0$ and by Eq. (9) it follows that $V'(\lambda_p) > 0$. By the previous argument we conclude that the sequence λ_n is increasing for $n > p$. Similarly, if $\lambda_p \leq 0$ then the sequnce is decreasing for $n > p$. In both cases the boundary condition (10) is violated and this proves (a).

The above argument shows that if $\lambda_{p+1} - \lambda_p > 0$ and $0 < \lambda_p < r$ then λ_n will be increasing for $n \geq p$, at least until $\lambda_n > r$. Hence, since $\lambda_n \to 0$, it follows that $\lambda_{n+1} - \lambda_n \leq 0$ for n large enough which establishes (b). This was also noted in [10].

From (b) and (9) we see that

$$S_N \equiv \sum_{n=1}^{N} V'(\lambda_n) \leq 0 \tag{11}$$

for N large enough. Since $\lambda_n \in (0, r]$ for n large we conclude that the sum S_N is increasing with N for N sufficiently large. Hence,

$$S_N \to S_\infty \equiv \sum_{n=0}^{\infty} V'(\lambda_n) \leq 0 \tag{12}$$

as $N \to \infty$. If $S_\infty = u < 0$ for some u then it follows from eq. (9) that

$$\lambda_{n+1} - \lambda_n \sim \frac{u}{2\theta} \frac{1}{n} \tag{13}$$

for n large. It follows that

$$\lambda_p - \lambda_q = \sum_{n=q}^{p-1} (\lambda_{n+1} - \lambda_n) \sim \frac{u}{2\theta} \sum_{n=q}^{p-1} \frac{1}{n} \to -\infty \tag{14}$$

for $p \to \infty$ contradicting $\lambda_p \to 0$ as $p \to \infty$. We conclude that $S_\infty = 0$ which also follows (formally) from the equation of motion by taking trace.

Since V' is approximately linear and increasing on a neighbourhood of 0 we conclude that the sums S_∞ and $\sum_n \lambda_n$ are absolutely convergent and the operator $\hat{\varphi}$ is trace class.

We remark that if $\hat{\varphi}$ is any solution to Eq. (5) then one can show by methods different from those of this paper that the spectrum of $\hat{\varphi}$ is contained in the interval $[0, s]$. This property will presumably be helpful for the study of non-spherically-symmetric solutions to Eq. (5).

4 Existence

In this section we use the results of the previous section and some elementary mathematical techniques to prove the existence of solutions to Eqs. (9) and (10). Let t be the location of the maximum of V' in the interval $[0, s]$ and let w be the location of the minimum of V' in the same interval.

Let us begin by assuming that

$$\theta \geq -\frac{2w}{V'(w)}. \tag{15}$$

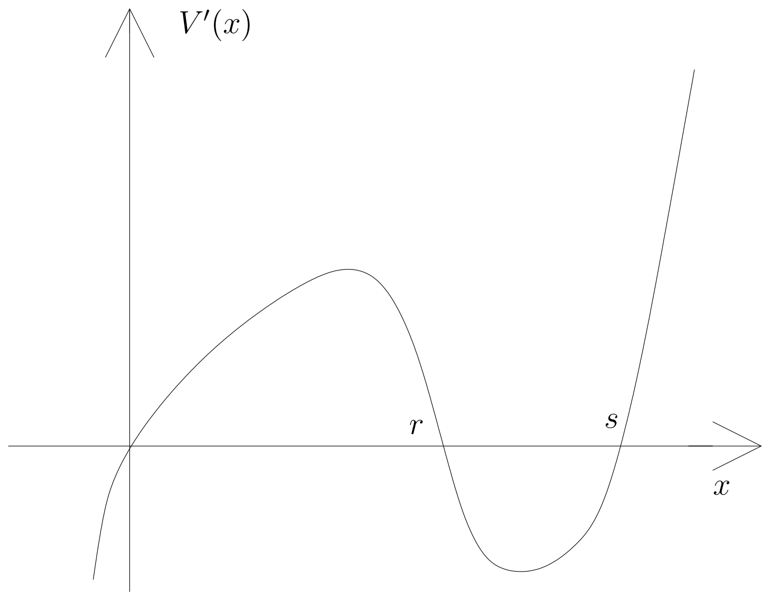

Figure 1: An graph of the derivative of a generic potential V which satisfies our assumptions.

In this case there is a unique largest $\underline{\lambda} \in [w, s)$ such that if we set $\lambda_0 = \underline{\lambda}$ in the recursion (9) then $\lambda_1 = 0$ and consequently $\lambda_n \to -\infty$ as $n \to \infty$. Evidently $\underline{\lambda}$ increases monotonically to s as $\theta \to \infty$. Now take $s > \lambda_0 > \underline{\lambda}$. Then $\lambda_1 > 0$. Assuming that θ is large enough we have

$$|V'(\underline{\lambda})| \leq V'(t) \tag{16}$$

and it follows that there is a unique $\bar{\lambda} \in (\underline{\lambda}, s)$ such that

$$V'(\lambda_1) = -V'(\lambda_0) \tag{17}$$

if $\lambda_0 = \bar{\lambda}$ and

$$V'(\lambda_1) < -V'(\bar{\lambda}) \tag{18}$$

for $\lambda_0 \in [\underline{\lambda}, \bar{\lambda})$. This means that for the sequence λ_n defined by $\lambda_0 = \bar{\lambda}$ and the recursion (9) we have $0 < \lambda_1 = \lambda_2 < \lambda_3$. If a sequence λ_n obeys the recursion (9) and has the property $\lambda_0 > \lambda_1 > \ldots > \lambda_N$ but $\lambda_{N+1} \geq \lambda_N$ we say that the sequence *turns at* N. Note that in this case $\lambda_N > 0$ by the proof of property (a) in Sec. 3. Furthermore, if $\lambda_{N+1} = \lambda_N$ then $\lambda_{N+2} > \lambda_{N+1}$.

Let us define the set

$$A = \{\lambda_0 \in [\underline{\lambda}, \bar{\lambda}] : \lambda_n \ \text{ turns at some } \ N\}. \tag{19}$$

By construction $\underline{\lambda} \notin A$ and $\bar{\lambda} \in A$. Put $\Lambda_0 = \inf A$. Since each λ_n depends continuously on the initial value λ_0 it follows that $\Lambda_0 < \bar{\lambda}$ and $\Lambda_0 \notin A$.

Now consider the sequence defined by $\lambda_0 = \Lambda_0$ and Eq. (9). Since this sequence does not turn it is monotonically decreasing. In order to show that this sequence provides a solution to our problem it therefore suffices to show that the sequence converges to 0. Suppose the sequence becomes negative for some n and let $n = N$ be the smallest value of n such that $\lambda_N < 0$. Then by the proof of property (a) in Sec. 3 we find that $\lambda_n \to -\infty$. By the continuity of λ_n as a function of λ_0 it follows that for λ_0 sufficiently close to Λ_0 the sequence λ_n converges monotonically to $-\infty$ but this contradicts the definition of Λ_0 as the infimum of those λ_0-values for which the sequence turns. It thus follows that the limit $\lim_{n\to\infty} \lambda_n = a \geq$ exists and by Eq. (9) we have

$$V'(a) = \frac{2}{\theta} \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0. \tag{20}$$

Hence, $a = 0$ since $\lambda_n \leq \lambda_1 < r$ by construction. This completes the proof of the existence of solutions for large θ.

The solutions whose existence was established above converge in the limit $\theta \to \infty$ to the projector on the ground state of the harmonic oscillator, corresponding to $\lambda_0 = s$ and $\lambda_n = 0$ for $n \geq 1$. We can easily generalize the above construction to obtain different spherically symmetric solutions. For example, we can tune the initial value λ_0 so that $\lambda_0, \lambda_1, \ldots, \lambda_n \in (r, s)$ but $\lambda_{n+1} \in (0, r)$ and the sequence is decreasing as before. In the infinite θ limit this solution converges to the projector on the subspace spanned by the first n harmonic oscillator states. Working slightly harder one can construct sequences with λ_0 close to 0 which jump to the interval $[r, s]$ at some N and thereafter converge monotonically to zero. These solutions correspond to a projection on the Nth eigenstate of the harmonic oscillator.

5 Nonexistence

In this section we will show that for sufficiently small θ the kinetic terms in the equation of motion, i.e., the left hand side of Eq. (7) cannot match the potential terms on the right hand side. Assume that the sequence λ_n is a solution to Eqs. (9)

and (10). If we multiply Eq. (7) by $\lambda_{n+1} - \lambda_{n-1}$ it can be written as

$$\begin{aligned}&(n+1)(\lambda_{n+1} - \lambda_n)^2 - n(\lambda_n - \lambda_{n-1})^2 + (\lambda_{n+1} - \lambda_n)(\lambda_n - \lambda_{n-1}) \\ &= \frac{\theta}{2} V'(\lambda_n)(\lambda_{n+1} - \lambda_n) + \frac{\theta}{2} V'(\lambda_n)(\lambda_n - \lambda_{n-1}).\end{aligned} \tag{21}$$

Let us first assume that λ_n is a decreasing sequence (as is the case for the solutions constructed in the previous section). Then we sum Eq. (21) over n from 1 to ∞. Defining $\Delta\lambda_n = (\lambda_{n+1} - \lambda_n)$ the result can be written

$$-\Delta\lambda_0^2 + \sum_{n=1}^{\infty} \Delta\lambda_n \Delta\lambda_{n-1} = \frac{\theta}{2} \sum_{n=1}^{\infty} V'(\lambda_n)(\Delta\lambda_n + \Delta\lambda_{n-1}). \tag{22}$$

We note by Eq. (9) and property (a) in Sec. 3 that

$$|\Delta\lambda_n| \leq c\theta \tag{23}$$

where the constant c depends only on the potential V but is independent of θ. It follows that the right hand side of Eq. (22) can be written as

$$\theta \int_{\lambda_0}^{0} V'(\lambda)\, d\lambda + O(\theta^2) = -\theta V(\lambda_0) + O(\theta^2). \tag{24}$$

We have

$$\sum_{n=1}^{\infty} \Delta\lambda_n \Delta\lambda_{n-1} \geq 0 \tag{25}$$

so Eq. (22) implies

$$0 \leq -\theta V(\lambda_0) + O(\theta^2). \tag{26}$$

By property (c) of solutions to Eq. (9) we know that $\lambda_0 > r$. It follows that the inequality (26) cannot be valid for small θ and we have a contradiction.

If the eigenvalues λ_n do not form a monotonic sequence we use property (b) of solutions and let N be such that $\lambda_n \leq \lambda_{n-1}$ for $n \geq N$ but $\lambda_{N-1} > \lambda_{N-2}$. Then $V'(\lambda_{N-1}) \leq 0$ so $\lambda_{N-1} \geq r$. Then we can repeat the argument above by summing Eq. (21) over n from N to ∞ and noting that

$$|\lambda_N - \lambda_{N-1}| \leq \frac{\theta}{2N} |V'(\lambda_{N-1})|. \tag{27}$$

This completes the proof of the fact that for sufficiently small θ (depending on the potential V) there do not exist any solutions to Eqs. (9) and (10).

6 Higher dimensions

If space has dimension $2d$, $d > 1$, we can choose coordinates such that the antisymmetric commutativity matrix is of a 2 by 2 block diagonal form where each block is identical to the noncommutativity matrix $\theta\varepsilon_{ij}$ in two dimensions. The operators corresponding to the functions on $\mathbf{R}^{2d}$ now act on a Hilbert space which can be regarded as the Hilbert space of d independent simple harmonic oscillators with raising and lowering operators a_i, a_i^*, $i = 1, \ldots d$. As a basis in this space we choose the vectors

$$|\mathbf{n}\rangle = |n_1 \ldots n_d\rangle \tag{28}$$

which are joint eigenvectors of the number operators $a_i^* a_i$ with nonnegative integer eigenvalues n_i, $i = 1 \ldots d$. It is readily checked that rotationally invariant functions on $\mathbf{R}^{2d}$ correspond to operators of the form

$$\hat{\varphi} = \sum_{\mathbf{n}} \lambda_n |\mathbf{n}\rangle\langle\mathbf{n}| \tag{29}$$

where the eigenvalues λ_n only depend on

$$n \equiv \sum_{i=1}^{d} n_i. \tag{30}$$

The equation of motion is now

$$2\sum_{i=1}^{d} [a_i, [a_i^*, \hat{\varphi}]] + \theta V'(\hat{\varphi}) = 0. \tag{31}$$

For operators of the form (29) we obtain from Eq. (31) the following genralization of Eq. (7)

$$(n + d)\lambda_{n+1} - (2n + d)\lambda_n + n\lambda_{n-1} = \frac{\theta}{2} V'(\lambda_n). \tag{32}$$

In order to derive a first order finite difference equation for the eigenvalues it is convenient to multiply Eq. (32) by a positive coefficient α_n chosen such that when we sum the equation over n from 0 to N all the λ_n's cancel except for λ_N and λ_{N-1}. In order for this to happen we must have

$$\alpha_{n+1} = \frac{n + d}{n + 1} \alpha_n \tag{33}$$

so if we choose $\alpha_0 = 1$ then

$$\alpha_n = \frac{(n + d) \ldots (n + 2)}{(d - 1)!}. \tag{34}$$

The generalization of Eq. (9) is therefore

$$\lambda_{N+1} - \lambda_N = \frac{\theta}{2(N+1)\alpha_N} \sum_{n=1}^{N} \alpha_n V'(\lambda_n). \tag{35}$$

For this recursion formula all the arguments of Secs. 3 and 4 go through with minor modifications. The same applies to the nonexistence proof for small values of θ which is now obtained by summing Eq. (32), multiplied on both sides by $\lambda_{n+1} - \lambda_{n-1}$, over n from the last turning point N to ∞ and observing that inequalities $|\Delta\lambda_n| \leq c\theta$ and $|\Delta\lambda_{N-1}| \leq c\theta |V'(\lambda_{N-1})|/N$ are still valid.

7 Discussion

In this paper we have established the existence of spherically symmetric solitons in all sufficiently strongly noncommutative even dimensional scalar field theories with potentials which increase at infinity and have two unequal minima. Clearly these solutions can be translated to generate other solutions. The existence and/or nonexistence (depending on θ) of solitons which are not rotationally invariant about any point is still an open problem.

The noncommutativity is responsible for the existence of these solitons which do not exist in the corresponding commutative or weakly noncommutative theories. The assumption of spherical symmetry simplifies the analysis but is presumably not necessary to prove nonexistence at small θ. At the present time we can generalize the nonexistence theorem of Derrick [9] to the weakly noncommutative setting under some regularity assumptions. This as well as the stability of solutions will be treated in a forthcoming paper.

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